INITIAL PHASE OF DEVELOPING DYNAMIC PERTURBATIONS IN A NONLINEARLY HEAT CONDUCTING GAS

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A model often used of radiative heat transport is the approximation of radiative heat conduction [1]. The validity condition of the model consists of small radiative mean free path in comparison with the characteristic size of the heated region.

A gas heated by radiation starts moving under the action of a radiative flux. To describe this process mathematically, along with the equation of radiative heat conductivity it is necessary to consider the system of equations of gas dynamics. It can be expected, however, that at the initial phase of evolving dynamic perturbations the radiative heat transfer process is practically independent of the gas motion (see, e.g., [2]). This fact makes it possible to analyze asymptotically the gas motion during its intense heating by radiation till the formation of isothermal rupture, which, as a result of further evolution is transformed into a shock wave, breaking away from the bulk of the gas heated by the radiation [1].

The system of equations of gas dynamics for one-dimensional motion of an ideal nonlinearly heat conducting gas is written in the following form in dimensionless variables

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}=0,\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial T}{\partial x}+\frac{T}{\rho}\frac{\partial\rho}{\partial x}=0,$$
$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+(\gamma-1)T\frac{\partial u}{\partial x}=\frac{1}{\rho}\frac{\partial}{\partial x}\lambda(T)\frac{\partial T}{\partial x}. \tag{1}$$

Here x is the coordinate along which heat is transported and the gas moves, t is time, $\rho(x, t)$, $u(x, t)$, and $T(x, t)$ are the gas density, velocity, and temperature, $\lambda(T)$ is the heat conduction coefficient of the gas, which in a substantial temperature interval can be written in the form $\lambda = nT^{n-1}$ $(n > 1)$ [1], and γ is the adiabat index.

Let the half-space $x > 0$ be filled by a cold $(T = 0)$, resting $(u = 0)$, and homogeneous $(\rho = 1)$ gas. Starting with the moment of time $t = 0$, the gas temperature at the boundary $x = 0$ varies by the power law $T = t^k$, and heat penetrates the region $x > 0$. It is assumed that the boundary $x = 0$ is impenetrable for the gas. Its motion is then completely described by system (1) with the boundary conditions

$$T = t,\ u = 0 \text{ for } x = 0; \tag{2}$$

$$T = \lambda\partial T/\partial x = 0 \text{ for } x = \infty \tag{3}$$

and the initial conditions

$$T = u = 0,\ \rho = 1 \text{ for } t = 0,\ x > 0. \tag{4}$$

It is well known that if the heat conductivity degenerates to $\lambda(0) = 0$, then heat in the cold gas propagates in the form of a thermal wave [3, 4], i.e., there exists a surface $x = x_f(t)$, strictly separating the region of the heated gas $(T > 0)$ $x < x_f(t)$ from the cold region $(T = 0)$ $x > x_f(t)$. In this case, due to the continuity of temperature and heat flux, the boundary conditions (3) must be satisfied at the surface $x = x_f(t)$ (the front of the thermal wave). Taking into account this feature of heat propagation in a nonlinearly heat conducting gas, as well as the initial and boundary conditions (2)-(4), we transform to the independent variables $x, t \to \eta = x/x_f(t)$, t, being "natural" for the process considered [2], and the dependent variables $T = t^k\theta(\eta, t)$, $u = t^\alpha v(\eta, t)$, $\rho = 1 + t^\beta r(\eta, t)$, $x_f = a(t)t^\sigma$. The constants $\alpha > 0$, $\beta > 0$, $\sigma > 0$ are subject to determination. Assuming that the functions $\theta(\eta, t)$, $v(\eta, t)$, $r(\eta, t)$, $a(t)$, along with their derivatives, have asymptotic values of order $O(1)$ for $t \to 0$, then the following transition to the new variables in system (1) they are found in the form $\alpha = \frac{1}{2}[k(3-n)+1]$, $\beta = k(2-n)+1$, $\sigma = \frac{1}{2}[k(n-1)+1]$. The system (1) is transformed in this case to the form

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$$\beta r - \sigma\eta\frac{\partial r}{\partial\eta} + \frac{1}{a}\frac{\partial v}{\partial\eta} + \frac{t^\beta}{a}\frac{\partial vr}{\partial\eta} + t\left[\frac{\partial r}{\partial t} - \frac{\dot a}{a}\eta\frac{\partial r}{\partial\eta}\right] = 0,$$

$$\alpha v - \sigma\eta\frac{\partial v}{\partial\eta} + \frac{1}{a}\frac{\partial\theta}{\partial\eta} + \frac{t^\beta}{a}\left[\frac{\theta}{1+t^\beta r}\frac{\partial r}{\partial\eta} + \frac{v}{a}\frac{\partial v}{\partial\eta}\right] + t\left[\frac{\partial v}{\partial t} - \frac{\dot a}{a}\frac{\partial v}{\partial\eta}\right] = 0, \tag{5}$$

$$\frac{1}{a^2(1+t^\beta r)}\frac{\partial^2\theta^n}{\partial\eta^2} - k\theta + \sigma\eta\frac{\partial\theta}{\partial\eta} + t^\beta\left[\frac{v}{a}\frac{\partial\theta}{\partial\eta} + (\gamma-1)\frac{\theta}{a}\frac{\partial v}{\partial\eta}\right] + t\left[\frac{\partial\theta}{\partial t} - \frac{\dot a}{a}\eta\frac{\partial\theta}{\partial\eta}\right] = 0.$$

It follows from (5) that for $t \to 0$ radiative heat transfer can be treated independently of the gas motion. The functions $\theta(\eta, t)$ and $a(t)$ are independent of time, and are determined from the equation

$$\frac{1}{a^2}\frac{\partial^2\theta^n}{\partial\eta^2} + \sigma\eta\frac{\partial\theta}{\partial\eta} - k\theta = 0 \tag{6}$$

and boundary conditions being a consequence of (2), (3):

$$\theta = 1 \text{ for } \eta = 0, \quad \theta = \frac{\partial\theta}{\partial\eta} = 0 \text{ for } \eta = 1. \tag{7}$$

The solution of problem (6), (7) can be carried out either numerically or found analytically in the form of series in powers of $(1 - \eta)$. Within the same approximation the functions $r(\eta, t) = r(\eta)$ and $v(\eta, t) = v(\eta)$ are obtained in the form of quadratures (see [2]):

$$v = \eta^{\frac{\alpha}{\sigma}}\frac{1}{a\sigma}\int_1^\eta \eta^{-\left(1+\frac{\alpha}{\sigma}\right)}\frac{\partial\theta}{\partial\eta}d\eta,$$
$$r = \eta^{\frac{\beta}{\sigma}}\frac{1}{a\sigma}\int_1^\eta \eta^{-\left(1+\frac{\beta}{\sigma}\right)}\frac{\partial v}{\partial\eta}d\eta \tag{8}$$

(here we took into account the nature of gas motion near the front of the thermal wave [5]). Applying l'Hôspital's rule, we have from (8)

$$v(0) = \lim_{\eta\to 0} v(\eta) = -\frac{1}{a\alpha}\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} \neq 0, \tag{9}$$

and the second boundary condition of (2) seems unsatisfied, with its nonsatisfaction in the asymptotic representation of the solution implying that near the boundary $\eta = 0$ there exists a thin boundary layer, in which the function $v(\eta, t)$ varies from a value $O(1)$ at the exterior boundary layer to zero at $\eta = 0$ [6, 7]. The width of this boundary layer can be determined by comparing the law of boundary motion of the thermal wave $x_f(t) \sim t^\sigma$ with the propagation law of sound perturbations near the boundary $x = 0$: $x_v(t) \sim t^{(k+2)/2}$. Consequently, the solutions (8) can be refined within the limits $\eta \sim t^{\beta/2}$.

For this, according to the scheme of [6, 7], we transform in (5) to the new independent variable $\eta^* = \eta/\varepsilon^{\beta/2}$ and to new dependent dynamic variables $r^* = \varepsilon^{-\delta}r$, $v^* = \varepsilon^{-\omega}v$, putting $\varepsilon = O(t)$. Analysis of the system of equations thus obtained shows that $\delta = -\beta/2$, $\omega = 0$. Comparing this system with (5) and putting $\beta/2 < 1$, one can write the asymptotic shape of Eq. (5), describing uniformly in $0 < \eta < 1$ the process with accuracy up to quantities of order $O(t^\varphi)$, where $\varphi = \min\{\beta, 1\}$. The functions θ, a are primarily found from (6), (7), while the following relations are obtained for determining $r(\eta, t)$ and $v(\eta, t)$

$$\chi(\eta, t)\frac{\partial r}{\partial\eta} = \frac{1}{a}(1+t^\beta r)\left(\alpha r + \frac{1}{a}\frac{\partial\theta}{\partial\eta}\right) - \beta\left(\frac{t^\beta}{a}v - \sigma\eta\right)r,$$

$$\chi(\eta, t)\frac{\partial v}{\partial\eta} = \frac{\beta}{a}\frac{\theta t^\beta}{1+t^\beta r}r - \left(\frac{t^\beta}{a}r - \sigma\eta\right)\left(\alpha v + \frac{1}{a}\frac{\partial\theta}{\partial\eta}\right)\left(\chi(\eta, t) = \left(-\sigma\eta + \frac{t^\beta}{a}v\right)^2 - \frac{\theta}{a^2}t^\beta\right). \tag{10}$$

Equations (10) must be solved for the boundary conditions

$$v = 0,\ r = 0 \text{ for } \eta = 1,\ v = 0 \text{ for } \eta = 0, \tag{11}$$

which are a consequence of (2)-(4) [5].

Here it is necessary to note that the asymptotic shape of the equations of motion of a nonlinearly heat conducting gas (6), (10) is valid when the inequalities $\sigma > 0$, $0 < \beta < 2$ are satisfied, which, as indicated, were used in deriving relations (6), (10). These inequalities determine the range of possible values of the constants k, n: $-1 < k(n - 2) < 1$, $k(n - 1) > -1$. It is also noted that for $\beta = 0$ the problem has an invariant solution, investigated in [8].

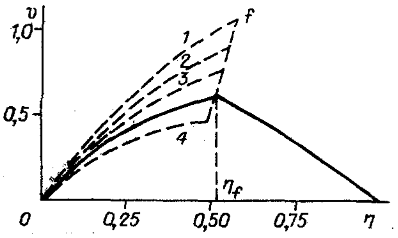

Fig. 1

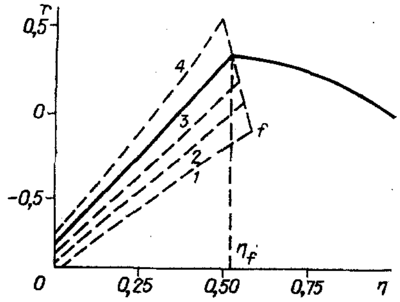

Fig. 2

The system (10) has a singular point, whose position is given by the relation $\eta = \eta_f(t)$, where the function $\eta_f(t)$ is determined from the equation $\chi(\eta_f, t) = 0$. The character of the singular point for $t \to 0$ can be investigated by methods of the qualitative theory [9]. The analysis provided showed that for $t \to 0$ it is a node. Therefore it seems possible to construct continuous solutions of system (10) satisfying conditions (11), though their number exceeds the order of system (10). For $t > 0$ the behavior of the integral curve of system (10) can be found only numerically.

For numerical analysis it is convenient to transform from (10) to the equivalent dynamic system

$$\frac{\partial r}{\partial \tau} = \frac{1}{a}\left(1 + t^\beta r\right)\left(\alpha r + \frac{1}{a}\frac{\partial \theta}{\partial \eta}\right) - \beta\left(\frac{t^\beta}{a} v - \sigma\eta\right) r,$$
$$\frac{\partial v}{\partial \tau} = \frac{\beta}{a}\frac{\theta t^\beta}{1 + t^\beta r} r - \left(\frac{t^\beta}{a} r - \sigma\eta\right)\left(\alpha v + \frac{1}{a}\frac{\partial \theta}{\partial \eta}\right), \quad \frac{\partial \eta}{\partial \tau} = \chi(\eta, t). \tag{12}$$

Here τ is a new variable along the integral curve in the variable space $\Omega(\eta, v, r)$. It must be noted that the limiting transition $\eta \to \eta_f$ corresponds to $\tau \to -\infty$. Therefore, to construct the numerical solution in the whole range of definition $0 < \eta < 1$ it is necessary to treat two Cauchy problems for the dynamic system (12) with different initial conditions following from (11): $\eta = 1$, $r = v = 0$ and $\eta = v = 0$, $r = r^0$. Since the constant r^0 is unknown ahead of time, the second Cauchy problem determines, generally speaking, a single-parameter family of integral curves, forming some surface Ω_1 in the variable space Ω. The constant r^0 is found from the matching condition of the solution of both Cauchy problems at the singular point $\tau \to -\infty$.

The indicated Cauchy problems were integrated by the modified Euler method. The step in the independent variable τ was taken equal to $\Delta\tau = -0.1$, as well as $\Delta_1\tau = \Delta\tau/3$. In Figs. 1 and 2 we show as examples for $k = 0$ (in the actual work over this special case we had the participation of K. A. Volosov), $n = 2$, $t = 0.3$ the projections of the integral curves thus constructed (the solid lines). The dashed lines 1-4 are the projections of several integral curves for $r(0, t) \neq r^0$. The lines in Figs. 1 and 2, denoted by identical digits, correspond to projections of one and the same integral curve. The line f is the projection of lines of singular points, confining the surface Ω_1.

This character of integral curves is retained only during a finite time. For $t > t_*(k, n) > 0$ the integral curve near the singular point is "reversed," and a convolution is formed on it [10]. An example of such an integral curve in projection near the singular point is shown in Figs. 3 and 4 for $k = 0$, $n = 2$, $t = 0.7$ (the notations coincide with Figs. 1 and 2). Starting from physical considerations, it is necessary to treat only single-valued solutions of the boundary-value problem (10), (11). Therefore, for $t > t_*(k, n)$ the solution must contain a strong discontinuity which, generally sepaking, transforms with a jump from one integral curve to the other.

Let the jump be observed for $\eta = \eta_s(t)$. The values of the gas parameters up to the jump $v|_{\eta_s+0} = v_+$, $r|_{\eta_s+0} = r_+$ are then uniquely related to their values following the jump $v|_{\eta_s-0} = v_-$, $r|_{\eta_s-0} = r_-$. These conditions in the approximation considered are obtained directly from Eq. (10). Integrating (10) over the limits from $\eta_s - \xi$ to $\eta_s + \xi$ ($\xi > 0$), and then passing to the limit $\xi \to 0$, we obtain

$$a\sigma\eta_s[r] - [v] - t^\beta[v, r] = 0,$$
$$a\alpha\eta_s[v] - (t^\beta/2)[v^2] - \theta_s[\ln(1 + t^\beta r)] = 0, \tag{13}$$

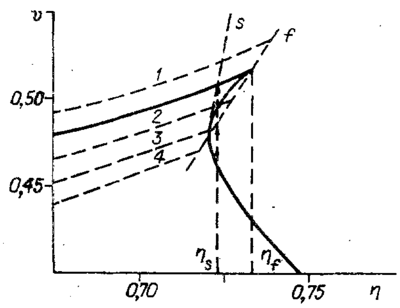

Fig. 3

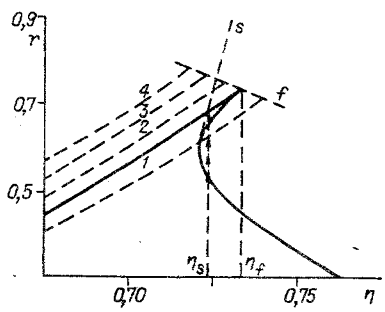

Fig. 4

where $\theta_s = \theta|_{\eta=\eta_s}$, and the square brackets denote the jump value of the function within them. It is appropriate to recall that within the approximation adopted the function $\theta(\eta)$ remains continuous for $\eta = \eta_s$ along with its derivatives.

It is natural to assume that immediately following its generation the jump is small ($[v], [r] \ll 1$). Expanding, then, the logarithm in a series, and taking into account terms in $[v]$ and $[r]$ from [13], we have

$$[v] = \frac{2}{t^\beta}\varkappa(\varkappa^2 - t^\beta\theta_s)(\varkappa^2 + t^\beta\theta_s)^{-1},$$
$$[r] = [v](1 + t^\beta r_+)(\varkappa^2 - t^\beta[v])^{-1} \quad (\varkappa = a\sigma\eta_s - t^\beta v_+). \tag{14}$$

Assigning the position of the jump $\eta_s(t)$, we determine v_+ and r_+ from the corresponding branch of the integral curve of problem (10), (11), while v_- and r_- are found by means of relationship (14). Since $\eta_s(t)$ is unknown ahead of time, all possible r_-, v_-, η_s values form a line s in the variable space Ω. The intersection point of this line with the surface Ω_1 determines the jump position. In Figs. 3 and 4 the dashed lines denote the projections of the line s. Numerical calculations have shown that within the calculation accuracy the jump within the approximation adopted is realized on the integral curve which is the continuous solution of the boundary-value problem (the solid curve in Figs. 3 and 4). The position and value of the jump in dynamic variables are shown in Figs. 3 and 4 by arrows.

It is necessary to note that the analyzed nature of developing dynamic perturbations for k = 0, n = 2 occurs in the whole region of variation of parameters k, n indicated above. Thus, we have established the generation regime of an isothermal breakup during intense thermal action on an ideal nonlinearly heat conducting gas. The strong breakup in the dynamic variable is generated as a result of evolution of a weak breakup, existing only for a finite time.

LITERATURE CITED

1. Ya. B. Zel'dovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Academic Press, New York (1967).
2. V. P. Shidlovskii, "Evolution of dynamic perturbations at the initial phase of point discontinuity in a heat conducting gas," Prikl. Mekh. Tekh. Fiz., No. 1 (1978).
3. Ya. B. Zel'dovich and A. S. Kompaneets, "The theory of heat distribution for a temperature-dependent heat conductivity," in: A. F. Ioffe's 70th Birthday [in Russian], Izd. Akad. Nauk SSSR, Moscow (1950).
4. O. A. Oleinik, A. S. Kalashnikov, and Yui-Lin' Chou, "Cauchy problems and boundary-value problems for equations of the type of nonstationary filtration," Izv. Akad. Nauk SSSR, Ser. Mat., 22, No. 5 (1958).
5. L. D. Pokrovskii and S. N. Taranenko, "Features of wave propagation in gas dynamics with nonlinear heat conduction," Tr. MVTU, No. 374 (1982).
6. W. R. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Interscience, New York (1965).
7. A. B. Vasil'eva and V. F. Butuzov, Asymptotic Expansions of Solutions of Singular Perturbation Equations [in Russian], Nauka, Moscow (1973).
8. P. P. Volosevich, S. P. Kurdyumov, et al., "Solution of the one-dimensional planar problem of piston motion in an ideal heat conducting gas," Zh. Vychisl Mat. Mat. Fiz., No. 1 (1963).
9. S. Lefschetz, Differential Equations: Geometric Theory, Interscience, New York (1963).
10. T. Poston and I. Stewart, Catastrophe Theory and Its Applications, Pitman, London (1978).